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On Conjugates and Algebraic Forms of Bicomplex Numbers: Structures and Relationships

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Abstract:

This paper investigates the structure of bicomplex numbers, with an emphasis on various types of conjugates, algebraic operations between those conjugates, and representation of conjugates in three forms: real, complex, and idempotent. We explore the behaviour and nature of the conjugates of a *bicomplex number. We also deal with the similarities and differences between the conjugates of the idempotent components of a bicomplex number and the idempotent components of its conjugates, supported by relevant examples, theorems, and remarks.*

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Introduction:

Bicomplex numbers form a commutative extension of complex numbers and have found applications in functional analysis, geometry, and mathematical physics. In the study of bicomplex analysis, it is essential to recognize the fundamental algebraic contrast between bicomplex numbers and quaternions. The quaternion algebra is a division algebra, meaning that every nonzero element admits a multiplicative inverse, and no zero divisors exist. In contrast, the bicomplex number system forms a commutative algebra that contains zero divisors and an infinite set of singular (non-invertible) elements. This distinction significantly influences both the algebraic and analytic behaviour of functions defined over these structures[3, 4].

This paper focuses on the multiple conjugates of bicomplex numbers and their algebraic behavior under standard operations. An important aspect of bicomplex analysis lies in the study of conjugations and their interaction with standard algebraic operations such as addition, multiplication, and subtraction. Unlike in the complex domain, where a single involutive conjugation suffices, bicomplex numbers admit multiple types of conjugation, each exhibiting unique algebraic behavior. We aim to identify which identities from classical complex analysis persist in the bicomplex setting, and which fail or require modification. Our approach combines analytical reasoning with illustrative examples to highlight the distinct characteristics of conjugations in this extended algebraic framework. This study not only strengthens the structural understanding of bicomplex systems but also lays a foundation for their potential use in extended function theory and operator analysis.

Building upon these foundational ideas, the present paper is divided into four sections. The first section outlines the necessary preliminaries and notations, whereas the second and third sections develop the main theoretical results, supported by proofs and examples that reinforce the analytical framework. In the last section, we discuss some open problems.

1. Preliminaries and Notations:

In this section, we recall some basic definitions, notations, and preliminary results concerning bicomplex numbers, which will be useful in the subsequent analysis. These preliminaries provide the algebraic framework necessary for studying the conjugates and algebraic forms introduced later. The notations and conventions used in this paper are adopted consistently from the standard works on bicomplex analysis, particularly those of Price [6], Srivastava and Srivastava [12], Rochon and Tremblay [9, 10], and Luna-Elizarrarás et al. [5], unless stated otherwise. These sources provide a coherent algebraic framework for studying bicomplex numbers and their related structures. Let us recall the definition of bicomplex numbers.

Definition 1.1 (Bicomplex numbers [6]): A number ξ, which is in the form of $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$, where x_1, x_2, x_3, and x_4 are real numbers, $i_1^2 = -1 = i_2^2$, and $i_1 i_2 = i_2 i_1$, is said to be a bicomplex number.

When a bicomplex number ξ is expressed in the above form, it is called the real form of ξ. In addition to this, the bicomplex number ξ can also be represented in another form, known as the complex form of ξ, given by $\xi = z_1 + i_2 z_2$, where z_1 and z_2 are i_1- complex numbers. The symbols $\mathbb{C}_2$, $\mathbb{C}_1$, and $\mathbb{C}_0$ denote the set of all bicomplex numbers, the set of all complex numbers, and the set of all real numbers, respectively.

Thus

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$$

and

$$\mathbb{C}_2 = \{x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

$$= \{z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\}.$$

The structure $(\mathbb{C}_2, +, \cdot, \times)$ forms an algebra over $\mathbb{C}_1$ and $\mathbb{C}_0$, but it does not constitute a field, since it contains divisors of zero.The algebra $\mathbb{C}_2$ possesses exactly four idempotent elements, 0, 1, $\frac{1+i_1i_2}{2}$, and $\frac{1-i_1i_2}{2}$. Among these, $\frac{1+i_1i_2}{2}$ and $\frac{1-i_1i_2}{2}$ are nontrivial idempotent elements, denoted by e_1 and e_2, respectively. Moreover, $\mathbb{C}_2$ is a commutative algebra with unity 1 (see[7]). The idempotent elements, e_1 and e_2, satisfy the following fundamental properties.

(a) $e_1 + e_2 = 1$.

(b) $e_1e_2 = 0$.

These idempotent elements play a fundamental role in the algebraic decomposition of bicomplex numbers, enabling expressions to be written in idempotent form. Every bicomplex number $\xi = z_1 + i_2z_2$ can be expressed uniquely as the complex combination of e_1 and e_2 as,

$$\xi = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2.$$

The complex combinations $(z_1 - i_1z_2)$ and $(z_1 + i_1z_2)$ are said to be idempotent components of ξ, denoted by $\xi^-({}^1\xi)$ and $\xi^+({}^2\xi)$, respectively (see [2, 11, 12]).

Thus $\xi = \xi^-e_1 + \xi^+e_2$ (or $\xi = {}^1\xi e_1 + {}^2\xi e_2$). This is the third form of representation of a bicomplex number ξ, known as the idempotent representation of ξ. The other two forms of ξ have already been discussed earlier. Therefore

$$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 \leftrightarrow \text{Real form of } \xi.$$

$$= z_1 + i_2z_2 \leftrightarrow \text{Complex form of } \xi.$$

$$= \xi^-e_1 + \xi^+e_2 \leftrightarrow \text{Idempotent representation of } \xi.$$

It is evident that complex numbers and hyperbolic numbers arise as special cases of bicomplex numbers. Specifically, when $x_3 = 0 = x_4$, the bicomplex number reduces to a complex number, whereas when $x_2 = 0 = x_3$, it reduces to a hyperbolic number.The algebraic properties of bicomplex numbers and their special cases are equally important and useful, as they provide valuable insights into the structural characteristics and internal consistency of the bicomplex framework (see [8]).

Remark 1.2: The operations addition (+), scalar multiplication ($\cdot$), and multiplication ($\times$) are perfectly compatible with idempotent components of bicomplex numbers and the scalars i.e., if $\xi = z_1 + i_2 z_2$, $\eta = w_1 + i_2 w_2$ and $\alpha \in \mathbb{C}_1$, then

$$\xi + \eta = (\xi^- + \eta^-)e_1 + (\xi^+ + \eta^+)e_2,$$

$$\xi \times \eta = (\xi^- \eta^-)e_1 + (\xi^+ \eta^+)e_2,$$

$$\alpha \cdot \xi = (\alpha\xi^-)e_1 + (\alpha\xi^+)e_2.$$

Moreover, $\xi \times \eta = (z_1 + i_2 z_2) \times (w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2(z_1 w_2 + z_2 w_1)$.

Definition 1.3 (Some special subsets of $\mathbb{C}_2$): The following subsets of $\mathbb{C}_2$ are necessary to mention to understand the nature of this paper.

(a) $H = \{x_1 + i_1 i_2 x_2 : x_1, x_2 \in \mathbb{R}\}$.

(b) $\mathbb{C}(i_1) = \{x_1 + i_1 x_2 : x_1, x_2 \in \mathbb{R}\}$.

(c) $\mathbb{C}(i_2) = \{x_1 + i_2 x_2 : x_1, x_2 \in \mathbb{R}\}$.

(d) $i_1\mathbb{C}(i_2) = \{i_1\xi : \xi \in \mathbb{C}(i_2)\}$.

(e) $i_2\mathbb{C}(i_1) = \{i_2\xi : \xi \in \mathbb{C}(i_1)\}$.

Moreover, the element of $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$ are said to be i_1-complex number and i_2-complex number, respectively.

Definition 1.4 (Conjugate of a bicomplex number): Analogous to the concept of the conjugate of a number in $\mathbb{C}_1$, there are three types of conjugates of a bicomplex number $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$. The i_1-conjugate, i_2-conjugate, and $i_1 i_2$-conjugate are denoted by $\overline{\xi}$, $\widetilde{\xi}$, and $\xi^\#$, respectively, and defined as follows:

$$\overline{\xi} = (x_1 - i_1 x_2) + i_2(x_3 - i_1 x_4),$$

$$\widetilde{\xi} = (x_1 + i_1 x_2) - i_2(x_3 + i_1 x_4),$$

$$\xi^\# = (x_1 - i_1 x_2) - i_2(x_3 - i_1 x_4).$$

It is very easy to see that $\bar{\xi} = (\bar{z}_1 + i_2\bar{z}_2) = \overline{(\xi^+)}e_1 + \overline{(\xi^-)}e_2$, $\tilde{\xi} = (z_1 - i_2 z_2) = (\xi^+)e_1 + (\xi^-)e_2$, and $\xi^{\#} = (\bar{z}_1 - i_2\bar{z}_2) = \overline{(\xi^-)}e_1 + \overline{(\xi^+)}e_2$. Moreover, if $\eta = y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4 = w_1 + i_2 w_2 \in \mathbb{C}_2$ then

$$\overline{(\xi \times \eta)} = \bar{\xi} \times \bar{\eta},$$
$$\widetilde{(\xi \times \eta)} = \tilde{\xi} \times \tilde{\eta},$$
$$(\xi \times \eta)^{\#} = \xi^{\#} \times \eta^{\#}, \text{ (see[1])}.$$

2. Algebra of Bicomplex Numbers with its Conjugates:

In this section, we examine the algebraic structure of bicomplex numbers in conjunction with their various conjugates. We investigate the behaviour of a bicomplex number when it is operated with its conjugates under the standard algebraic operations of addition, subtraction, and multiplication. This analysis provides deeper insight into the internal consistency of the bicomplex system and highlights the distinctive properties that differentiate it from classical complex algebra.

Additionally, we analyse the compatibility of the idempotent components of a bicomplex number with different types of conjugations and also examine the mutual compatibility among the conjugates themselves. This comparison plays a crucial role in understanding how conjugation interacts with conjugation itself and with idempotent decomposition, further clarifying the structural consistency of bicomplex algebra.

Proposition 2.1: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then

(a) The sum of the number ξ with its i_1-conjugate is an element of the set $\mathbb{C}(i_2)$, i.e. $\xi + \bar{\xi} \in \mathbb{C}(i_2)$.

(b) $\xi + \bar{\xi} = 2(\text{Real part of } z_1) + i_2 2(\text{Real part of } z_2)$.

(c) $(\xi + \bar{\xi})^- = \xi^- + \overline{\xi^+}$ and $(\xi + \bar{\xi})^+ = \xi^+ + \overline{\xi^-}$.

(d) The difference of the number ξ and its i_1-conjugate is an element of the set $i_1\mathbb{C}(i_2)$.

(e) $\xi - \overline{\xi} = 2i_1$(Imaginary part of z_1) $+ i_2\, 2i_1$(Imaginary part of z_2).

(f) $(\xi - \overline{\xi})^- = \xi^- - \overline{\xi^+}$ and $(\xi - \overline{\xi})^+ = \xi^+ - \overline{\xi^-}$.

Proof: Since,

$$\begin{aligned}\xi + \overline{\xi} &= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) + (x_1 - i_1x_2 + i_2x_3 - i_1i_2x_4)\\ &= 2(x_1 + i_2x_3)\\ &= (z_1 + i_2z_2) + (\overline{z}_1 + i_2\overline{z}_2), \text{ by (1.4)}\\ &= (z_1 + \overline{z}_1) + i_2(z_2 + \overline{z}_2)\\ &= 2(\text{Real part of } z_1) + i_2\, 2(\text{Real part of } z_2)\\ &= \left(\xi^- e_1 + \xi^+ e_2\right) + \left(\overline{\xi^+} e_1 + \overline{\xi^-} e_2\right), \text{ by (1.4)}\\ &= \left(\xi^- + \overline{\xi^+}\right)e_1 + \left(\xi^+ + \overline{\xi^-}\right)e_2.\end{aligned}$$

Similarly,

$$\begin{aligned}\xi - \overline{\xi} &= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) - (x_1 - i_1x_2 + i_2x_3 - i_1i_2x_4)\\ &= 2(i_1x_2 + i_1i_2x_4)\\ &= i_1(2x_2 + i_2 2x_4)\\ &= (z_1 + i_2z_2) - (\overline{z}_1 + i_2\overline{z}_2), \text{ by (1.4)}\\ &= (z_1 - \overline{z}_1) + i_2(z_2 - \overline{z}_2)\\ &= 2i_1(\text{Imaginary part of } z_1) + i_2\, 2i_1(\text{Imaginary part of } z_2)\\ &= \left(\xi^- e_1 + \xi^+ e_2\right) - \left(\overline{\xi^+} e_1 + \overline{\xi^-} e_2\right), \text{ by (1.4)}\\ &= \left(\xi^- - \overline{\xi^+}\right)e_1 + \left(\xi^+ - \overline{\xi^-}\right)e_2.\end{aligned}$$

This completes the proof of the proposition.

Remark 2.2: It is evident that if $\eta = y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4 = w_1 + i_2w_2 \in \mathbb{C}_2$, and any one of the following holds:

(a) $\xi + \eta = 2(x_1 + i_2x_3)$.

(b) $\xi + \eta = 2(\text{Real part of } z_1) + i_2\, 2(\text{Real part of } z_2)$.

(c) $(\xi + \eta)^- = \xi^- + \overline{\xi^+}$ and $(\xi + \eta)^+ = \xi^+ + \overline{\xi^-}$.

Then η is the i_1-conjugate of ξ.

Proposition 2.3: Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then

(a) The sum of the number ξ with its i_2-conjugate is an element of the set $\mathbb{C}(i_1)$, i.e. $\xi + \tilde{\xi} \in \mathbb{C}(i_1)$.

(b) $\xi + \tilde{\xi} = 2z_1$.

(c) $\left(\xi + \tilde{\xi}\right)^- = \xi^- + \xi^+$ and $\left(\xi + \tilde{\xi}\right)^+ = \xi^+ + \xi^-$.

(d) The difference of the number ξ and its i_2-conjugate is an element of the set $i_2\mathbb{C}(i_1)$.

(e) $\xi - \tilde{\xi} = i_2 2z_2$.

(f) $\left(\xi - \tilde{\xi}\right)^- = \xi^- - \xi^+$ and $\left(\xi - \tilde{\xi}\right)^+ = \xi^+ - \xi^-$.

Proof: Since,

$$\begin{aligned}
\xi + \tilde{\xi} &= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) + (x_1 + i_1x_2 - i_2x_3 - i_1i_2x_4)\\
&= 2(x_1 + i_1x_2)\\
&= (z_1 + i_2z_2) + (z_1 - i_2z_2), \text{ by (1.4)}\\
&= 2z_1\\
&= \left(\xi^- e_1 + \xi^+ e_2\right) + \left(\xi^+ e_1 + \xi^- e_2\right), \text{ by (1.4)}\\
&= \left(\xi^- + \xi^+\right) e_1 + \left(\xi^+ + \xi^-\right) e_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\xi - \tilde{\xi} &= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) - (x_1 + i_1x_2 - i_2x_3 - i_1i_2x_4)\\
&= 2(i_2x_3 + i_1i_2x_4)\\
&= i_2(2x_3 + i_12x_4)
\end{aligned}$$

$$= (z_1 + i_2 z_2) - (z_1 - i_2 z_2), \text{ by } (1.4)$$
$$= i_2 2 z_2$$
$$= \left(\xi^- e_1 + \xi^+ e_2\right) - \left(\xi^+ e_1 + \xi^- e_2\right), \text{ by } (1.4)$$
$$= \left(\xi^- - \xi^+\right) e_1 + \left(\xi^+ - \xi^-\right) e_2.$$

This completes the proof of the proposition.

Remark 2.4: It is evident that if $\eta = y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4 = w_1 + i_2 w_2 \in \mathbb{C}_2$, and any one of the following holds:

(a) $\xi + \eta = 2(x_1 + i_1 x_2)$.

(b) $\xi + \eta = 2z_1$.

(c) $(\xi + \eta)^- = \xi^- + \xi^+$ and $(\xi + \eta)^+ = \xi^+ + \xi^-$.

Then η is the i_2-conjugate of ξ.

Proposition 2.5: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then

(a) The sum of the number ξ with its $i_1 i_2$-conjugate is an element of the set H, i.e. $\xi + \xi^{\#} \in H$.

(b) $\xi + \xi^{\#} = 2(\text{Real part of } z_1) + i_2 2 i_1(\text{Imaginary part of } z_2)$.

(c) $\left(\xi + \xi^{\#}\right)^- = \xi^- + \overline{\xi^-}$ and $\left(\xi + \xi^{\#}\right)^+ = \xi^+ + \overline{\xi^+}$.

(d) The difference of the number ξ and its $i_1 i_2$-conjugate is the sum of i_1-complex number and i_2-complex number.

(e) $\xi - \xi^{\#} = 2 i_1(\text{Imaginary part of } z_1) + i_2 2(\text{Real part of } z_2)$.

(f) $\left(\xi - \xi^{\#}\right)^- = \xi^- - \overline{\xi^-}$ and $\left(\xi - \xi^{\#}\right)^+ = \xi^+ - \overline{\xi^+}$.

Proof: Since,

$$\xi + \xi^{\#} = (x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) + (x_1 - i_1 x_2 - i_2 x_3 + i_1 i_2 x_4)$$
$$= 2(x_1 + i_1 i_2 x_4)$$

$$= (z_1 + i_2 z_2) + (\bar{z}_1 - i_2 \bar{z}_2), \text{ by (1.4)}$$
$$= (z_1 + \bar{z}_1) + i_2(z_2 - \bar{z}_2)$$
$$= 2(\text{Real part of } z_1) + i_2\, 2i_1(\text{Imaginary part of } z_2)$$
$$= \left(\xi^- e_1 + \xi^+ e_2\right) + \left(\overline{\xi^-} e_1 + \overline{\xi^+} e_2\right), \text{ by (1.4)}$$
$$= \left(\xi^- + \overline{\xi^-}\right)e_1 + \left(\xi^+ + \overline{\xi^+}\right)e_2.$$

Similarly,

$$\xi - \xi^{\#} = (x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) - (x_1 - i_1 x_2 - i_2 x_3 + i_1 i_2 x_4)$$
$$= 2(i_1 x_2 + i_2 x_3)$$
$$= (z_1 + i_2 z_2) - (\bar{z}_1 - i_2 \bar{z}_2), \text{ by (1.4)}$$
$$= (z_1 - \bar{z}_1) + i_2(z_2 + \bar{z}_2)$$
$$= 2i_1(\text{Imaginary part of } z_1) + i_2 2(\text{Real part of } z_2)$$
$$= \left(\xi^- e_1 + \xi^+ e_2\right) - \left(\overline{\xi^-} e_1 + \overline{\xi^+} e_2\right), \text{ by (1.4)}$$
$$= \left(\xi^- - \overline{\xi^-}\right)e_1 + \left(\xi^+ - \overline{\xi^+}\right)e_2.$$

This completes the proof of the proposition.

Remark 2.6: It is evident that if $\eta = y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4 = w_1 + i_2 w_2 \in \mathbb{C}_2$, and any one of the following holds:

(a) $\xi + \eta = 2(x_1 + i_1 i_2 x_4)$.

(b) $\xi + \eta = 2(\text{Real part of } z_1) + i_2\, 2i_1(\text{Imaginary part of } z_2)$.

(c) $(\xi + \eta)^- = \xi^- + \overline{\xi^-}$ and $(\xi + \eta)^+ = \xi^+ + \overline{\xi^+}$.

Then η is the $i_1 i_2$-conjugate of ξ.

Proposition 2.7: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then

(a) The sum of the i_1-conjugate and i_2-conjugate of the number ξ is an element of the set H, i.e. $\bar{\xi} + \tilde{\xi} \in H$.

(b) $\overline{\xi} + \widetilde{\xi} = 2(\text{Real part of } z_1) - i_2\, 2i_1(\text{Imaginary part of } z_2)$.

(c) $\left(\overline{\xi} + \widetilde{\xi}\right)^- = \overline{\xi^+} + \xi^+$ and $\left(\overline{\xi} + \widetilde{\xi}\right)^+ = \overline{\xi^-} + \xi^-$.

(d) The difference of the i_1-conjugate and i_2-conjugate is the sum of i_1-complex number and i_2-complex number.

(e) $\overline{\xi} - \widetilde{\xi} = -2i_1(\text{Imaginary part of } z_1) + i_2\, 2(\text{Real part of } z_2)$.

(f) $\left(\overline{\xi} - \widetilde{\xi}\right)^- = \overline{\xi^+} - \xi^+$ and $\left(\overline{\xi} - \widetilde{\xi}\right)^+ = \overline{\xi^-} - \xi^-$.

Proof: Since,

$$\begin{aligned}
\overline{\xi} + \widetilde{\xi} &= (x_1 - i_1x_2 + i_2x_3 - i_1i_2x_4) + (x_1 + i_1x_2 - i_2x_3 - i_1i_2x_4)\\
&= 2(x_1 - i_1i_2x_4)\\
&= (\overline{z}_1 + i_2\overline{z}_2) + (z_1 - i_2z_2), \text{ by (1.4)}\\
&= (z_1 + \overline{z}_1) + i_2(\overline{z}_2 - z_2)\\
&= 2(\text{Real part of } z_1) - i_2\, 2i_1(\text{Imaginary part of } z_2)\\
&= \left(\overline{\xi^+}e_1 + \overline{\xi^-}e_2\right) + \left(\xi^+e_1 + \xi^-e_2\right), \text{ by (1.4)}\\
&= \left(\overline{\xi^+} + \xi^+\right)e_1 + \left(\overline{\xi^-} + \xi^-\right)e_2
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{\xi} - \widetilde{\xi} &= (x_1 - i_1x_2 + i_2x_3 - i_1i_2x_4) - (x_1 + i_1x_2 - i_2x_3 - i_1i_2x_4)\\
&= 2(-i_1x_2 + i_2x_3)\\
&= (\overline{z}_1 + i_2\overline{z}_2) - (z_1 - i_2z_2), \text{ by (1.4)}\\
&= (\overline{z}_1 - z_1) + i_2(\overline{z}_2 + z_2)\\
&= -2i_1(\text{Imaginary part of } z_1) + i_2\, 2(\text{Real part of } z_2)\\
&= \left(\overline{\xi^+}e_1 + \overline{\xi^-}e_2\right) - \left(\xi^+e_1 + \xi^-e_2\right), \text{ by (1.4)}\\
&= \left(\overline{\xi^+} - \xi^+\right)e_1 + \left(\overline{\xi^-} - \xi^-\right)e_2.
\end{aligned}$$

This completes the proof of the proposition.

Proposition 2.8: Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then

(a) The sum of the i_1-conjugate and i_1i_2-conjugate of the number ξ is an element of the set $\mathbb{C}(i_1)$, i.e. $\overline{\xi} + \xi^{\#} \in \mathbb{C}(i_1)$.

(b) $\overline{\xi} + \xi^{\#} = (2\overline{z_1})$.

(c) $\left(\overline{\xi} + \xi^{\#}\right)^{-} = \overline{\xi^{+}} + \overline{\xi^{-}}$ and $\left(\overline{\xi} + \xi^{\#}\right)^{+} = \overline{\xi^{-}} + \overline{\xi^{+}}$.

(d) The difference of the i_1-conjugate and i_1i_2-conjugate of the number ξ is an element of the set $i_2\mathbb{C}(i_1)$.

(e) $\overline{\xi} - \xi^{\#} = i_2 2(\overline{z_2})$.

(f) $\left(\overline{\xi} - \xi^{\#}\right)^{-} = \overline{\xi^{+}} - \overline{\xi^{-}}$ and $\left(\overline{\xi} - \xi^{\#}\right)^{+} = \overline{\xi^{-}} - \overline{\xi^{+}}$.

Proof: Since,

$$\begin{aligned}
\overline{\xi} + \xi^{\#} &= (x_1 - i_1x_2 + i_2x_3 - i_1i_2x_4) + (x_1 - i_1x_2 - i_2x_3 + i_1i_2x_4)\\
&= 2(x_1 - i_1x_2)\\
&= (\overline{z_1} + i_1\overline{z_2}) + (\overline{z_1} - i_2\overline{z_2}), \text{ by } (1.4)\\
&= 2(\overline{z_1})\\
&= \left(\overline{\xi^{+}}e_1 + \overline{\xi^{-}}e_2\right) + \left(\overline{\xi^{-}}e_1 + \overline{\xi^{+}}e_2\right), \text{ by } (1.4)\\
&= \left(\overline{\xi^{+}} + \overline{\xi^{-}}\right)e_1 + \left(\overline{\xi^{-}} + \overline{\xi^{+}}\right)e_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{\xi} - \xi^{\#} &= (x_1 - i_1x_2 + i_2x_3 - i_1i_2x_4) - (x_1 - i_1x_2 - i_2x_3 + i_1i_2x_4)\\
&= 2(i_2x_3 - i_1i_2x_4)\\
&= i_2(2x_3 - i_12x_4)\\
&= (\overline{z_1} + i_2\overline{z_2}) - (\overline{z_1} - i_2\overline{z_2}), \text{ by } (1.4)\\
&= i_2 2(\overline{z_2})\\
&= \left(\overline{\xi^{+}}e_1 + \overline{\xi^{-}}e_2\right) - \left(\overline{\xi^{-}}e_1 + \overline{\xi^{+}}e_2\right), \text{ by } (1.4)
\end{aligned}$$

$$= \left(\overline{\xi^+} - \overline{\xi^-}\right)e_1 + \left(\overline{\xi^-} - \overline{\xi^+}\right)e_2.$$

This completes the proof of the proposition.

Proposition 2.9: Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then

(a) The sum of the i_2-conjugate and i_1i_2-conjugate of the number ξ is an element of the set $\mathbb{C}(i_2)$, i.e. $\tilde{\xi} + \xi^\# \in \mathbb{C}(i_2)$.

(b) $\tilde{\xi} + \xi^\# = 2(\text{Real part of } z_1) - i_2\, 2(\text{Real part of } z_2)$.

(c) $\left(\tilde{\xi} + \xi^\#\right)^- = \xi^+ + \overline{\xi^-}$ and $\left(\tilde{\xi} + \xi^\#\right)^+ = \xi^- + \overline{\xi^+}$.

(d) The difference of the i_2-conjugate and i_1i_2-conjugate of the number ξ is an element of the set $i_1\mathbb{C}(i_2)$.

(e) $\tilde{\xi} - \xi^\# = 2i_1(\text{Imaginary part of } z_1) - i_2\, 2i_1(\text{Imaginary part of } z_2)$.

(f) $\left(\tilde{\xi} - \xi^\#\right)^- = \xi^+ - \overline{\xi^-}$ and $\left(\tilde{\xi} - \xi^\#\right)^+ = \xi^- - \overline{\xi^+}$.

Proof: Since,

$$\begin{aligned}
\tilde{\xi} + \xi^\# &= (x_1 + i_1x_2 - i_2x_3 - i_1i_2x_4) + (x_1 - i_1x_2 - i_2x_3 + i_1i_2x_4) \\
&= 2(x_1 - i_2x_3) \\
&= (z_1 - i_2z_2) + (\bar{z}_1 - i_2\bar{z}_2), \text{ by (1.4)} \\
&= (z_1 + \bar{z}_1) - i_2(z_2 + \bar{z}_2) \\
&= 2(\text{Real part of } z_1) - i_2\, 2(\text{Real part of } z_2) \\
&= \left(\xi^+e_1 + \xi^-e_2\right) + \left(\overline{\xi^-}e_1 + \overline{\xi^+}e_2\right), \text{ by (1.4)} \\
&= \left(\xi^+ + \overline{\xi^-}\right)e_1 + \left(\xi^- + \overline{\xi^+}\right)e_2
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{\xi} - \xi^\# &= (x_1 + i_1x_2 - i_2x_3 - i_1i_2x_4) - (x_1 - i_1x_2 - i_2x_3 + i_1i_2x_4) \\
&= 2(i_1x_2 - i_1i_2x_4) \\
&= i_1(2x_2 - i_2 2x_4)
\end{aligned}$$

$$= (z_1 - i_2 z_2) - (\bar{z}_1 - i_2 \bar{z}_2), \text{ by } (1.4)$$
$$= (z_1 - \bar{z}_1) - i_2(z_2 - \bar{z}_2)$$
$$= 2i_1(\text{Imaginary part of } z_1) - i_2\, 2i_1(\text{Imaginary part of } z_2)$$
$$= \left(\xi^+ e_1 + \xi^- e_2\right) - \left(\overline{\xi^-} e_1 + \overline{\xi^+} e_2\right), \text{ by } (1.4)$$
$$= \left(\xi^+ - \overline{\xi^-}\right) e_1 + \left(\xi^- - \overline{\xi^+}\right) e_2.$$

This completes the proof of the proposition.

Proposition 2.10: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then

(a) The multiplication of the number ξ with its i_1-conjugate is an element of the set $\mathbb{C}(i_2)$, i.e. $\xi \times \bar{\xi} \in \mathbb{C}(i_2)$.

(b) $\xi \times \bar{\xi} = |z_1|^2 - |z_2|^2 + i_2(z_1 \bar{z}_2 + z_2 \bar{z}_1)$.

(c) $\left(\xi \times \bar{\xi}\right)^- = \xi^- \overline{\xi^+}$ and $\left(\xi \times \bar{\xi}\right)^+ = \overline{\left(\xi \times \bar{\xi}\right)^-}$.

Proof: Since,

$$\xi \times \bar{\xi} = (x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) \times (x_1 - i_1 x_2 + i_2 x_3 - i_1 i_2 x_4)$$
$$= \{(x_1 + i_2 x_3) + (i_1 x_2 + i_1 i_2 x_4)\} \times \{(x_1 + i_2 x_3) - (i_1 x_2 + i_1 i_2 x_4)\}$$
$$= (x_1 + i_2 x_3)^2 - (i_1 x_2 + i_1 i_2 x_4)^2$$
$$= x_1^2 - x_3^2 + i_2 2 x_1 x_3 + x_2^2 - x_4^2 + i_2 2 x_2 x_4$$
$$= x_1^2 + x_2^2 - x_3^2 - x_4^2 + i_2(2 x_1 x_3 + 2 x_2 x_4)$$
$$= (z_1 + i_2 z_2) \times (\bar{z}_1 + i_2 \bar{z}_2), \text{ by } (1.4)$$
$$= z_1 \bar{z}_1 + i_2 z_1 \bar{z}_2 + i_2 z_2 \bar{z}_1 - z_2 \bar{z}_2$$
$$= z_1 \bar{z}_1 - z_2 \bar{z}_2 + i_2(z_1 \bar{z}_2 + z_2 \bar{z}_1)$$
$$= |z_1|^2 - |z_2|^2 + i_2(z_1 \bar{z}_2 + z_2 \bar{z}_1)$$
$$= \left(\xi^- e_1 + \xi^+ e_2\right) \times \left(\overline{\xi^+} e_1 + \overline{\xi^-} e_2\right), \text{ by } (1.4)$$

$$= \left(\xi^- \overline{\xi^+}\right) e_1 + \left(\xi^+ \overline{\xi^-}\right) e_2$$
$$= \left(\xi^- \overline{\xi^+}\right) e_1 + \left(\overline{\xi^- \overline{\xi^+}}\right) e_2.$$

This completes the proof of the proposition.

Proposition 2.11: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then

(a) The multiplication of the number ξ with its i_2-conjugate is an element of the set $\mathbb{C}(i_1)$, i.e. $\xi \times \tilde{\xi} \in \mathbb{C}(i_1)$.

(b) $\xi \times \tilde{\xi} = z_1^2 + z_2^2$.

(c) $\left(\xi \times \tilde{\xi}\right)^- = \xi^- \xi^+$ and $\left(\xi \times \tilde{\xi}\right)^+ = \left(\xi \times \tilde{\xi}\right)^-$.

Proof: Since,

$$\begin{aligned}
\xi \times \tilde{\xi} &= (x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) \times (x_1 + i_1 x_2 - i_2 x_3 - i_1 i_2 x_4) \\
&= \{(x_1 + i_1 x_2) + (i_2 x_3 + i_1 i_2 x_4)\} \times \{(x_1 + i_1 x_2) - (i_2 x_3 + i_1 i_2 x_4)\} \\
&= (x_1 + i_1 x_2)^2 - (i_2 x_3 + i_1 i_2 x_4)^2 \\
&= x_1^2 - x_2^2 + i_1 2 x_1 x_2 + x_3^2 - x_4^2 + i_2 2 x_3 x_4 \\
&= x_1^2 - x_2^2 + x_3^2 - x_4^2 + i_1 (2 x_1 x_2 + 2 x_3 x_4) \\
&= (z_1 + i_2 z_2) \times (z_1 - i_2 z_2), \text{ by (1.4)} \\
&= z_1^2 + z_2^2 \\
&= \left(\xi^- e_1 + \xi^+ e_2\right) \times \left(\xi^+ e_1 + \xi^- e_2\right), \text{ by (1.4)} \\
&= \left(\xi^- \xi^+\right) e_1 + \left(\xi^+ \xi^-\right) e_2.
\end{aligned}$$

This completes the proof of the proposition.

Proposition 2.12: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then

(a) The multiplication of the number ξ with its $i_1 i_2$-conjugate is an element of the set H, i.e. $\xi \times \xi^{\#} \in H$.

(b) $\xi \times \xi^{\#} = |z_1|^2 + |z_2|^2 - i_2(z_1\bar{z}_2 - z_2\bar{z}_1)$.

(c) $\left(\xi \times \xi^{\#}\right)^- = |\xi^-|^2$ and $\left(\xi \times \xi^{\#}\right)^+ = |\xi^+|^2$.

Proof: Since,

$$\begin{aligned}
\xi \times \xi^{\#} &= (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) \times (x_1 - i_1x_2 - i_2x_3 + i_1i_2x_4)\\
&= \{(x_1 + i_1i_2x_4) + (i_1x_2 + i_2x_3)\} \times \{(x_1 + i_1i_2x_4) - (i_1x_2 + i_2x_3)\}\\
&= (x_1 + i_1i_2x_4)^2 - (i_1x_2 + i_2x_3)^2\\
&= x_1^2 + x_4^2 + i_1i_2 2x_1x_4 + x_2^2 + x_3^2 - i_1i_2 2x_2x_3\\
&= x_1^2 + x_2^2 + x_3^2 + x_4^2 + i_1i_2(2x_1x_4 - 2x_2x_3)\\
&= (z_1 + i_2z_2) \times (\bar{z}_1 - i_2\bar{z}_2), \text{ by (1.4)}\\
&= z_1\bar{z}_1 - i_2z_1\bar{z}_2 + i_2z_2\bar{z}_1 + z_2\bar{z}_2\\
&= z_1\bar{z}_1 + z_2\bar{z}_2 - i_2(z_1\bar{z}_2 - z_2\bar{z}_1)\\
&= |z_1|^2 + |z_2|^2 - i_2(z_1\bar{z}_2 - z_2\bar{z}_1)\\
&= \left(\xi^- e_1 + \xi^+ e_2\right) \times \left(\overline{\xi^-} e_1 + \overline{\xi^+} e_2\right), \text{ by (1.4)}\\
&= \left(\xi^- \overline{\xi^-}\right)e_1 + \left(\xi^+ \overline{\xi^+}\right)e_2\\
&= |\xi^-|^2 e_1 + |\xi^+|^2 e_2.
\end{aligned}$$

This completes the proof of the proposition.

Proposition 2.13: Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then

(a) The multiplication of the i_1-conjugate and i_2-conjugate of the number ξ is an element of the set H, i.e. $\bar{\xi} \times \tilde{\xi} \in H$.

(b) $\bar{\xi} \times \tilde{\xi} = |z_1|^2 + |z_2|^2 - i_2(\bar{z}_1z_2 - \bar{z}_2z_1)$.

(c) $\left(\bar{\xi} \times \tilde{\xi}\right)^- = |\xi^+|^2$ and $\left(\bar{\xi} \times \tilde{\xi}\right)^+ = |\xi^-|^2$.

Proof: Since,

$$\begin{aligned}\overline{\xi} \times \widetilde{\xi} &= (x_1 - i_1x_2 + i_2x_3 - i_1i_2x_4) \times (x_1 + i_1x_2 - i_2x_3 - i_1i_2x_4)\\
&= \{(x_1 - i_1i_2x_4) + (i_2x_3 - i_1x_2)\} \times \{(x_1 - i_1i_2x_4) - (i_2x_3 - i_1x_2)\}\\
&= (x_1 - i_1i_2x_4)^2 - (i_2x_3 - i_1x_2)^2\\
&= x_1^2 + x_4^2 - i_1i_2 2x_1x_4 + x_3^2 + x_2^2 + i_1i_2 2x_3x_2\\
&= x_1^2 + x_2^2 + x_3^2 + x_4^2 - i_1i_2(2x_1x_4 - 2x_3x_2)\\
&= (\overline{z}_1 + i_2\overline{z}_2) \times (z_1 - i_2z_2), \text{ by (1.4)}\\
&= \overline{z}_1z_1 - i_2\overline{z}_1z_2 + i_2\overline{z}_2z_1 + \overline{z}_2z_2\\
&= |z_1|^2 + |z_2|^2 - i_2(\overline{z}_1z_2 - \overline{z}_2z_1)\\
&= \left(\overline{\xi^+}e_1 + \overline{\xi^-}e_2\right) \times \left(\xi^+e_1 + \xi^-e_2\right), \text{ by (1.4)}\\
&= \left(\overline{\xi^+}\,\xi^+\right)e_1 + \left(\overline{\xi^-}\,\xi^-\right)e_2\\
&= |\xi^+|^2\, e_1 + |\xi^-|^2\, e_2.\end{aligned}$$

This completes the proof of the proposition.

Proposition 2.14: Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then

(a) The multiplication of the i_1-conjugate and i_1i_2-conjugate of the number ξ is an element of the set $\mathbb{C}(i_1)$, i.e. $\overline{\xi} \times \xi^\# \in \mathbb{C}(i_1)$.

(b) $\overline{\xi} \times \xi^\# = \overline{z}_1^2 + \overline{z}_2^2$.

(c) $\left(\overline{\xi} \times \xi^\#\right)^- = \overline{\xi^+}\,\overline{\xi^-} = \left(\overline{\xi} \times \xi^\#\right)^+$.

Proof: Since,

$$\begin{aligned}\overline{\xi} \times \xi^\# &= (x_1 - i_1x_2 + i_2x_3 - i_1i_2x_4) \times (x_1 - i_1x_2 - i_2x_3 + i_1i_2x_4)\\
&= \{(x_1 - i_1x_2) + (i_2x_3 - i_1i_2x_4)\} \times \{(x_1 - i_1x_2) - (i_2x_3 - i_1i_2x_4)\}\\
&= (x_1 - i_1x_2)^2 - (i_2x_3 - i_1i_2x_4)^2\end{aligned}$$

$$= x_1^2 - x_2^2 - i_1 2x_1x_2 + x_3^2 - x_4^2 - i_1 2x_3x_4$$
$$= x_1^2 - x_2^2 + x_3^2 - x_4^2 - i_1(2x_1x_2 + 2x_3x_4)$$
$$= (\bar{z}_1 + i_2\bar{z}_2) \times (\bar{z}_1 - i_2\bar{z}_2), \text{ by } (1.4)$$
$$= \bar{z}_1^{\,2} + \bar{z}_2^2$$
$$= \left(\overline{\xi^+}e_1 + \overline{\xi^-}e_2\right) \times \left(\overline{\xi^-}e_1 + \overline{\xi^+}e_2\right), \text{ by } (1.4)$$
$$= \left(\overline{\xi^+}\,\overline{\xi^-}\right)e_1 + \left(\overline{\xi^-}\,\overline{\xi^+}\right)e_2$$
$$= \overline{\xi^+}\,\overline{\xi^-}.$$

This completes the proof of the proposition.

Proposition 2.15: Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then

(a) The multiplication of the i_2-conjugate and i_1i_2-conjugate of the number ξ is an element of the set $\mathbb{C}(i_2)$, i.e. $\tilde{\xi} \times \xi^{\#} \in \mathbb{C}(i_2)$.

(b) $\tilde{\xi} \times \xi^{\#} = |z_1|^2 - |z_2|^2 - i_2(z_1\bar{z}_2 + z_2\bar{z}_1)$.

(c) $\left(\tilde{\xi} \times \xi^{\#}\right)^- = \xi^+\,\overline{\xi^-}$ and $\left(\tilde{\xi} \times \xi^{\#}\right)^+ = \overline{\left(\tilde{\xi} \times \xi^{\#}\right)^-}$.

Proof: Since,

$$\tilde{\xi} \times \xi^{\#} = (x_1 + i_1x_2 - i_2x_3 - i_1i_2x_4) \times (x_1 - i_1x_2 - i_2x_3 + i_1i_2x_4)$$
$$= \{(x_1 - i_2x_3) + (i_1x_2 - i_1i_2x_4)\} \times \{(x_1 - i_2x_3) - (i_1x_2 - i_1i_2x_4)\}$$
$$= (x_1 - i_2x_3)^2 - (i_1x_2 - i_1i_2x_4)^2$$
$$= x_1^2 - x_3^2 - i_1 2x_1x_3 + x_2^2 - x_4^2 - i_2 2x_2x_4$$
$$= x_1^2 + x_2^2 - x_3^2 - x_4^2 - i_2(2x_1x_3 + 2x_2x_4)$$
$$= (z_1 - i_2z_2) \times (\bar{z}_1 - i_2\bar{z}_2), \text{ by } (1.4)$$
$$= z_1\bar{z}_1 - i_2z_1\bar{z}_2 - i_2z_2\bar{z}_1 - z_2\bar{z}_2$$
$$= |z_1|^2 - |z_2|^2 - i_2(z_1\bar{z}_2 - z_2\bar{z}_1)$$

$$= \left(\xi^{+} e_1 + \xi^{-} e_2\right) \times \left(\overline{\xi^{-}} e_1 + \overline{\xi^{+}} e_2\right), \text{ by (1.4).}$$

$$= \left(\xi^{+}\,\overline{\xi^{-}}\right) e_1 + \left(\xi^{-}\,\overline{\xi^{+}}\right) e_2.$$

This completes the proof of the proposition.

Now, we examine whether the conjugates of the idempotent components of a bicomplex number coincide with the idempotent components of the corresponding conjugates of that bicomplex number.This is illustrated in (2.16).

Remark 2.16: As immediate consequences of the (1.4), we have the following results in general.

(a) The i_1-conjugate of the first idempotent component of ξ, $\overline{\xi^{-}}$, does not coincide with the first idempotent component of the i_1-conjugate of ξ, $(\overline{\xi})^{-}$; that is, $(\overline{\xi})^{-} = \overline{\xi^{+}} \neq \overline{\xi^{-}}$.

(b) The i_1-conjugate of the second idempotent component of ξ, $\overline{\xi^{+}}$, does not coincide with the second idempotent component of the i_1-conjugate of ξ, $(\overline{\xi})^{+}$; that is, $(\overline{\xi})^{+} = \overline{\xi^{-}} \neq \overline{\xi^{+}}$.

(c) The i_2-conjugate of the first idempotent component of ξ, $\widetilde{\xi^{-}}$, does not coincide with the first idempotent component of the i_2-conjugate of ξ, $(\widetilde{\xi})^{-}$; that is, $(\widetilde{\xi})^{-} = \xi^{+} \neq \xi^{-} = \widetilde{\xi^{-}}$.

(d) The i_2-conjugate of the second idempotent component of ξ, $\widetilde{\xi^{+}}$, does not coincide with the second idempotent component of the i_2-conjugate of ξ, $(\widetilde{\xi})^{+}$; that is, $(\widetilde{\xi})^{+} = \xi^{-} \neq \xi^{+} = \widetilde{\xi^{+}}$.

(e) The $i_1 i_2$-conjugate of the first idempotent component of ξ, $(\xi^{-})^{\#}$, coincides with the first idempotent component of the $i_1 i_2$-conjugate of ξ, $(\xi^{\#})^{-}$; that is, $(\xi^{\#})^{-} = \xi^{-} = (\xi^{-})^{\#}$.

(f) The $i_1 i_2$-conjugate of the second idempotent component of ξ, $(\xi^{+})^{\#}$, coincides with the second idempotent component of the $i_1 i_2$-conjugate of ξ, $(\xi^{\#})^{+}$; that is, $(\xi^{\#})^{+} = \overline{\xi^{+}} = (\xi^{+})^{\#}$.

In the following (2.17), we investigate the interaction among the i_1-conjugate, i_2-conjugate, and $i_1 i_2$-conjugate of a bicomplex number.

Remark 2.17: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then we have

(a) $(\tilde{\bar{\xi}}) = (\widetilde{\bar{z}_1 + i_2 \bar{z}_2}) = \bar{z}_1 - i_2 \bar{z}_2 = \xi^{\#}$, and $(\overline{\tilde{\xi}}) = (\overline{z_1 - i_2 z_2}) = \bar{z}_1 - i_2 \bar{z}_2 = \xi^{\#}$. Hence, the i_1-conjugate of the i_2-conjugate of the bicomplex number ξ coincides with the i_2-conjugate of the i_1-conjugate of the bicomplex number ξ, and both are equal to $i_1 i_2$-conjugate of the bicomplex number ξ.

(b) $(\bar{\xi})^{\#} = (\bar{z}_1 + i_2 \bar{z}_2)^{\#} = z_1 - i_2 z_2 = \tilde{\xi}$, and $(\overline{\xi^{\#}}) = (\overline{\bar{z}_1 - i_2 \bar{z}_2}) = z_1 - i_2 z_2 = \tilde{\xi}$. Hence, the i_1-conjugate of the $i_1 i_2$-conjugate of the bicomplex number ξ coincides with the $i_1 i_2$-conjugate of the i_1-conjugate of the bicomplex number ξ, and both are equal to i_2-conjugate of the bicomplex number ξ.

(c) $(\tilde{\xi})^{\#} = (z_1 - i_2 z_2)^{\#} = \bar{z}_1 + i_2 \bar{z}_2 = \bar{\xi}$, and $(\widetilde{\xi^{\#}}) = (\widetilde{\bar{z}_1 - i_2 \bar{z}_2}) = \bar{z}_1 + i_2 \bar{z}_2 = \bar{\xi}$. Hence, the i_2-conjugate of the $i_1 i_2$-conjugate of the bicomplex number ξ coincides with the $i_1 i_2$-conjugate of the i_2-conjugate of the bicomplex number ξ, and both are equal to i_1-conjugate of the bicomplex number ξ.

Thus, it is evident that any two of the three conjugations i_1-conjugate, i_2-conjugate, and $i_1 i_2$-conjugate are applied successively to a bicomplex number, then the resulting operation coincides with the remaining third conjugation.

3. Non-Ordinary Conjugates of a Bicomplex Number and their Compatibility:

In this section, we define four additional conjugations of a bicomplex number. The previously introduced three conjugations i_1-conjugate, i_2-conjugate, and $i_1 i_2$-conjugate are henceforth called ordinary conjugates, while the newly defined ones are called non-ordinary conjugates.

We further analyse the interaction between the non-ordinary conjugates and all existing conjugations of a bicomplex number. Moreover, the compatibility of each conjugation, both ordinary and non-ordinary, with addition (+), and multiplication (×) is examined. Finally, we present open problems and possible directions for future research.

Definition 3.1 (Generalized i_1-conjugate): Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then the generalized i_1-conjugate of ξ is denoted by ξ^{i_1} and defined as

$$\xi^{i_1} = x_1 - i_1x_2 + i_2x_3 + i_1i_2x_4.$$

It is very easy to verify that $\xi^{i_1} = \overline{z_1} + i_2z_2$ and the idempotent representation of ξ^{i_1} is

$$\left[\overline{\left(\frac{\xi^- + \xi^+}{2}\right)} + \left(\frac{\xi^- - \xi^+}{2}\right)\right]e_1 + \left[\overline{\left(\frac{\xi^- + \xi^+}{2}\right)} + \left(\frac{\xi^+ - \xi^-}{2}\right)\right]e_2.$$

Definition 3.2 (Generalized i_2-conjugate): Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then the generalized i_2-conjugate of ξ is denoted by ξ^{i_2} and defined as

$$\xi^{i_2} = x_1 + i_1x_2 - i_2x_3 + i_1i_2x_4.$$

It is very easy to verify that $\xi^{i_2} = z_1 - i_2\overline{z_2}$ and the idempotent representation of ξ^{i_2} is

$$\left[\left(\frac{\xi^- + \xi^+}{2}\right) + \overline{\left(\frac{\xi^- - \xi^+}{2}\right)}\right]e_1 + \left[\left(\frac{\xi^- + \xi^+}{2}\right) + \overline{\left(\frac{\xi^+ - \xi^-}{2}\right)}\right]e_2.$$

Definition 3.3 (Generalized i_1i_2-conjugate): Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then the generalized i_1i_2-conjugate of ξ is denoted by $\xi^{i_1i_2}$ and defined as

$$\xi^{i_1i_2} = x_1 + i_1x_2 + i_2x_3 - i_1i_2x_4.$$

It is very easy to verify that $\xi^{i_1i_2} = z_1 + i_2\overline{z_2}$ and the idempotent representation of $\xi^{i_1i_2}$ is

$$\left[\left(\frac{\xi^- + \xi^+}{2}\right) + \overline{\left(\frac{\xi^+ - \xi^-}{2}\right)}\right]e_1 + \left[\left(\frac{\xi^- + \xi^+}{2}\right) + \overline{\left(\frac{\xi^- - \xi^+}{2}\right)}\right]e_2.$$

Definition 3.4 (Natural-conjugate): Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then the natural-conjugate of ξ is denoted by ξ^* and defined as

$$\xi^* = x_1 - i_1x_2 - i_2x_3 - i_1i_2x_4.$$

It is very easy to verify that $\xi^* = \overline{z_1} - i_2z_2$ and the idempotent representation of ξ^* is

$$\left[\left(\frac{\overline{\xi^- + \xi^+}}{2}\right) + \left(\frac{\xi^+ - \xi^-}{2}\right)\right]e_1 + \left[\left(\frac{\overline{\xi^- + \xi^+}}{2}\right) + \left(\frac{\xi^- - \xi^+}{2}\right)\right]e_2.$$

In the following (3.5), (3.6), (3.7), (3.8), (3.9), and (3.10), we investgate the interaction of non-ordinary conjugates with both ordinary and non-ordinary conjugates of a bicomplex number.

Remark 3.5: Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then we have

(a) $\left(\xi^{i_1i_2}\right)^{i_1i_2} = \xi.$

(b) $\overline{\left(\xi^{i_1i_2}\right)} = \left(\overline{\xi}\right)^{i_1i_2} = \xi^{i_1}.$

(c) $\left(\xi^{i_1i_2}\right)^{i_1} = \left(\xi^{i_1}\right)^{i_1i_2} = \overline{\xi}.$

(d) $\overline{\left(\xi^{i_1}\right)} = \left(\overline{\xi}\right)^{i_1} = \xi^{i_1i_2}.$

It is evident that any two of the three conjugations i_1-conjugate, generalized i_1-conjugate, and generalized i_1i_2-conjugate are applied successively to a bicomplex number, then the resulting operation coincides with the remaining third conjugation.

Remark 3.6: Let $\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = z_1 + i_2z_2$ be a bicomplex number, then we have

(a) $\widetilde{\left(\xi^{i_1i_2}\right)} = \left(\widetilde{\xi}\right)^{i_1i_2} = \xi^{i_2}.$

(b) $\left(\xi^{i_1i_2}\right)^{i_2} = \left(\xi^{i_2}\right)^{i_1i_2} = \widetilde{\xi}.$

(c) $\widetilde{\left(\xi^{i_2}\right)} = \left(\widetilde{\xi}\right)^{i_2} = \xi^{i_1 i_2}.$

It is evident that any two of the three conjugations i_2-conjugate, generalized i_2-conjugate, and generalized $i_1 i_2$-conjugate are applied successively to a bicomplex number, then the resulting operation coincides with the remaining third conjugation.

Remark 3.7: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then we have

(a) $\left(\xi^{i_1 i_2}\right)^{\#} = \left(\xi^{\#}\right)^{i_1 i_2} = \xi^{*}.$

(b) $\left(\xi^{i_1 i_2}\right)^{*} = \left(\xi^{*}\right)^{i_1 i_2} = \xi^{\#}.$

(c) $\left(\xi^{\#}\right)^{*} = \left(\xi^{*}\right)^{\#} = \xi^{i_1 i_2}.$

It is evident that any two of the three conjugations $i_1 i_2$-conjugate, generalized $i_1 i_2$-conjugate, and natural-conjugate are applied successively to a bicomplex number, then the resulting operation coincides with the remaining third conjugation.

Remark 3.8: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then we have

(a) $\left(\xi^{*}\right)^{*} = \xi.$

(b) $\overline{\left(\xi^{*}\right)} = \left(\overline{\xi}\right)^{*} = \xi^{i_2}.$

(c) $\left(\xi^{*}\right)^{i_2} = \left(\xi^{i_2}\right)^{*} = \overline{\xi}.$

(d) $\overline{\left(\xi^{i_2}\right)} = \left(\overline{\xi}\right)^{i_2} = \xi^{*}.$

It is evident that any two of the three conjugations i_1-conjugate, generalized i_2-conjugate, and natural-conjugate are applied successively to a bicomplex number, then the resulting operation coincides with the remaining third conjugation.

Remark 3.9: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then we have

(a) $\widetilde{\left(\xi^{*}\right)} = \left(\widetilde{\xi}\right)^{*} = \xi^{i_1}$.

(b) $\left(\xi^{*}\right)^{i_1} = \left(\xi^{i_1}\right)^{*} = \widetilde{\xi}$.

(c) $\widetilde{\left(\xi^{i_1}\right)} = \left(\widetilde{\xi}\right)^{i_2} = \xi^{*}$.

It is evident that any two of the three conjugations i_2-conjugate, generalized i_1-conjugate, and natural-conjugate are applied successively to a bicomplex number, then the resulting operation coincides with the remaining third conjugation.

Remark 3.10: Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2$ be a bicomplex number, then we have

(a) $\left(\xi^{i_1}\right)^{i_1} = \xi$

(b) $\left(\xi^{i_2}\right)^{i_2} = \xi$

(c) $\left(\xi^{i_1}\right)^{\#} = \left(\xi^{\#}\right)^{i_1} = \xi^{i_2}$

(d) $\left(\xi^{i_1}\right)^{i_2} = \left(\xi^{i_2}\right)^{i_1} = \xi^{\#}$

(e) $\left(\xi^{i_2}\right)^{\#} = \left(\xi^{\#}\right)^{i_2} = \xi^{i_1}$

It is evident that any two of the three conjugations $i_1 i_2$-conjugate, generalized i_1-conjugate, and generalized i_2-conjugate are applied successively to a bicomplex number, then the resulting operation coincides with the remaining third conjugation.

Table 1 illustrates the interactions among all ordinary and non-ordinary conjugates of a bicomplex number.

	$\overline{\xi}$	$\widetilde{\xi}$	$\xi^{\#}$	ξ^{i_1}	ξ^{i_2}	$\xi^{i_1 i_2}$	ξ^{*}
i_1-conjugate	ξ	$\xi^{\#}$	$\widetilde{\xi}$	$\xi^{i_1 i_2}$	ξ^{*}	ξ^{i_1}	ξ^{i_2}
i_2-conjugate	$\xi^{\#}$	ξ	$\overline{\xi}$	ξ^{*}	$\xi^{i_1 i_2}$	ξ^{i_2}	ξ^{i_1}
$i_1 i_2$-conjugate	$\widetilde{\xi}$	$\overline{\xi}$	ξ	ξ^{i_2}	ξ^{i_1}	ξ^{*}	$\xi^{i_1 i_2}$
Generalized i_1-conjugate	$\xi^{i_1 i_2}$	ξ^{*}	ξ^{i_2}	ξ	$\xi^{\#}$	$\overline{\xi}$	$\widetilde{\xi}$
Generalized i_2-conjugate	ξ^{*}	$\xi^{i_1 i_2}$	ξ^{i_1}	$\xi^{\#}$	ξ	$\widetilde{\xi}$	$\overline{\xi}$
Generalized $i_1 i_2$-conjugate	ξ^{i_1}	ξ^{i_2}	ξ^{*}	$\overline{\xi}$	$\widetilde{\xi}$	ξ	$\xi^{\#}$
Natural-conjugate	ξ^{i_2}	ξ^{i_1}	$\xi^{i_1 i_2}$	$\widetilde{\xi}$	$\overline{\xi}$	$\xi^{\#}$	ξ

Table 1 : Interaction between conjugates of a bicomplex number ξ.

In the table, all entries corresponding to a_{62}, a_{53}, a_{74}, a_{35}, a_{26}, a_{47}, and a_{18} are incorrectly written as $\xi^{\#}$. They should be corrected to ξ^{*}.

Now, we examine the compatibility of all conjugations with addition (+), and multiplication (×). This is illustrated in (3.11).

Remark 3.11: Let ξ and η be two bicomplex number, then we have the following results in general.

(a) $\overline{(\xi + \eta)} = \overline{\xi} + \overline{\eta}$.

(b) $\widetilde{(\xi + \eta)} = \widetilde{\xi} + \widetilde{\eta}$.

(c) $(\xi + \eta)^{\#} = \xi^{\#} + \eta^{\#}$.

(d) $(\xi + \eta)^{i_1} = \xi^{i_1} + \eta^{i_1}$.

(e) $(\xi + \eta)^{i_2} = \xi^{i_2} + \eta^{i_2}$.

(f) $(\xi + \eta)^{i_1 i_2} = \xi^{i_1 i_2} + \eta^{i_1 i_2}$.

(g) $(\xi + \eta)^{*} = \xi^{*} + \eta^{*}$.

(h) $\overline{(\xi \times \eta)} = \overline{\xi} \times \overline{\eta}$.

(i) $\widetilde{(\xi \times \eta)} = \widetilde{\xi} \times \widetilde{\eta}$.

(j) $(\xi \times \eta)^{\#} = \xi^{\#} \times \eta^{\#}$.

Example 3.12: Let ζ be a bicomplex number such that $\zeta = 1 + i_1 3 + i_2 4 + i_1 i_2 8$, then

$$\overline{\zeta} = (1 - i_1 3) + i_2(4 - i_1 8),$$

$$\widetilde{\zeta} = (1 + i_1 3) - i_2(4 + i_1 8),$$

$$\zeta^{\#} = (1 - i_1 3) - i_2(4 - i_1 8),$$

$$\zeta^{i_1} = (1 - i_1 3) + i_2(4 + i_1 8),$$

$$\zeta^{i_2} = (1 + i_1 3) - i_2(4 - i_1 8),$$

$$\zeta^{i_1 i_2} = (1 + i_1 3) + i_2(4 - i_1 8),$$

$$\zeta^{*} = (1 - i_1 3) - i_2(4 + i_1 8).$$

4. Open Problems and Future Directions:

We leave the following problem for future work.

Problem 4.1: Determine whether the $\left(\left(\left(\xi^{\alpha_1}\right)^{\alpha_2}\right)^{\alpha_3}\right)^{\alpha_4} = \xi$, where α_1, α_2, α_3, and $\alpha_4 \in \{i_1, i_2, i_1 i_2, {}^*\}$.

Problem 4.2: Determine whether the following identities hold for a bicomplex number ξ:

(a) $\left(\widetilde{(\bar{\xi})}\right)^{\#} = \xi$.

(b) $\left(\overline{(\tilde{\xi})}\right)^{\#} = \xi$.

(c) $\left(\widetilde{\overline{(\xi^{\#})}}\right) = \xi$.

(d) $\overline{\left(\widetilde{(\xi^{\#})}\right)} = \xi$.

(e) $\left(\widetilde{(\bar{\xi})^{\#}}\right) = \xi$.

(f) $\overline{\left((\tilde{\xi})^{\#}\right)} = \xi$.

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